Last Time: Vectors + Operations
Dot Product.

Prop (Properties of Vector Addition): Let $\vec{u},\vec{v},\vec{w} \in \mathbb{R}^n$ and let $b,c \in \mathbb{R}$.

① $\vec{0} + \vec{u} = \vec{u}$ ← zero vector is the identity for vector addition

pf: $(0,0,\cdots,0) + (u_1,u_2,\cdots,u_n)$

$= (0+u_1, 0+u_2, \cdots, 0+u_n) = (u_1,u_2,\cdots,u_n)$ $\boxed{\checkmark}$

② $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ ← Commutativity of vector addition.

pf: $(u_1,u_2,\cdots,u_n) + (v_1,v_2,\cdots,v_n)$

$= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$

$= (v_1+u_1, v_2+u_2, \cdots, v_n+u_n)$

$= (v_1,v_2,\cdots,v_n) + (u_1,u_2,\cdots,u_n)$ $\boxed{\checkmark}$

③ $\vec{u} + (\vec{v}+\vec{w}) = (\vec{u}+\vec{v}) + \vec{w}$ ← vector addition is associative.

pf: $(u_1,u_2,\cdots,u_n) + \big((v_1,v_2,\cdots,v_n) + (w_1,w_2,\cdots,w_n)\big)$

$= (u_1,u_2,\cdots,u_n) + (v_1+w_1, v_2+w_2, \cdots, v_n+w_n)$

$= \big(u_1+(v_1+w_1), u_2+(v_2+w_2), \cdots, u_n+(v_n+w_n)\big)$

$= \big((u_1+v_1)+w_1, (u_2+v_2)+w_2, \cdots, (u_n+v_n)+w_n\big)$

$= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n) + (w_1,w_2,\cdots,w_n)$

$= \big((u_1,u_2,\cdots,u_n) + (v_1,v_2,\cdots,v_n)\big) + (w_1,w_2,\cdots,w_n)$ $\boxed{\checkmark}$

④ $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$ ← (Scalar multiplication distributes over vector addition)

pf: $c\big((u_1,u_2,\cdots,u_n) + (v_1,v_2,\cdots,v_n)\big)$

$= c(u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$

$= (c(u_1+v_1), c(u_2+v_2), \cdots, c(u_n+v_n))$

$= (cu_1+cv_1, cu_2+cv_2, \cdots, cu_n+cv_n)$

$= (cu_1, cu_2, \cdots, cu_n) + (cv_1, cv_2, \cdots, cv_n)$

$= c(u_1, u_2, \cdots, u_n) + c(v_1, v_2, \cdots, v_n)$ $\boxed{\checkmark}$

⑤ $(b+c)\vec{u} = b\vec{u} + c\vec{u}$ ← ("Scalars act on vectors")

pf: $(b+c)(u_1, u_2, \cdots, u_n)$

$= ((b+c)u_1, (b+c)u_2, \cdots, (b+c)u_n)$

$= (bu_1+cu_1, bu_2+cu_2, \cdots, bu_n+cu_n)$

$= (bu_1, bu_2, \cdots, bu_n) + (cu_1, cu_2, \cdots, cu_n)$

$= b(u_1, u_2, \cdots, u_n) + c(u_1, u_2, \cdots, u_n)$ $\boxed{\checkmark}$

⑥ $0\vec{u} = \vec{0}$ and $1\vec{u} = \vec{u}$ ← 0 and 1 act right...

pf: $0(u_1, u_2, \cdots, u_n) = (0u_1, 0u_2, \cdots, 0u_n) = (0, 0, \cdots, 0)$

$1(u_1, u_2, \cdots, u_n) = (1u_1, 1u_2, \cdots, 1u_n) = (u_1, u_2, \cdots, u_n)$ $\boxed{\checkmark}$

Recall: For $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
commutativity

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
dot product distributes over vector addition

③ $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
linearity

④ $\vec{0} \cdot \vec{u} = 0$
zero absorption

⑤ $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$
norm-squared law

Algebraic properties of the Dot Product.

Prop (Cauchy-Schwarz Inequality): Let $\vec{u},\vec{v} \in \mathbb{R}^n$.

Then $$|\vec{u}\cdot\vec{v}| \leq |\vec{u}||\vec{v}|$$

Pf: $$0 \leq \big|\, |\vec{v}|\vec{u} - |\vec{u}|\vec{v}\big|^2$$
$$= \big(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\big)\cdot\big(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\big)$$
$$= \big(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\big)\cdot|\vec{v}|\vec{u} - \big(|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\big)\cdot|\vec{u}|\vec{v}$$
$$= |\vec{v}|\vec{u}\cdot|\vec{v}|\vec{u} - |\vec{u}|\vec{v}\cdot|\vec{v}|\vec{u} - |\vec{v}|\vec{u}\cdot|\vec{u}|\vec{v} + |\vec{u}|\vec{v}\cdot|\vec{u}|\vec{v}$$
$$= |\vec{v}|^2\underbrace{(\vec{u}\cdot\vec{u})}_{|\vec{u}|^2} - |\vec{u}||\vec{v}|(\vec{v}\cdot\vec{u}) - |\vec{v}||\vec{u}|(\vec{u}\cdot\vec{v}) + |\vec{u}|^2\underbrace{(\vec{v}\cdot\vec{v})}_{|\vec{v}|^2}$$
$$= 2|\vec{u}|^2|\vec{v}|^2 - 2|\vec{u}||\vec{v}|(\vec{u}\cdot\vec{v})$$
$$= 2|\vec{u}||\vec{v}|\big(|\vec{u}||\vec{v}| - \vec{u}\cdot\vec{v}\big)$$

On the other hand $2|\vec{u}||\vec{v}| > 0$, so $|\vec{u}||\vec{v}| - \vec{u}\cdot\vec{v} \geq 0$.

Hence $\vec{u}\cdot\vec{v} \leq |\vec{u}||\vec{v}|$ as desired ☐

Remark: I skipped the case $2|\vec{u}||\vec{v}| = 0$, because this implies either $|\vec{u}| = 0$ or $|\vec{v}| = 0$ (and thus $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$).

Prop (Triangle Inequality):

If $\vec{u},\vec{v} \in \mathbb{R}^n$, then $|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$.

NB: Let's consider vectors $\vec{u} = (1,2,3)$ and $\vec{v} = (-3,1,0)$.

$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{(-3)^2+1^2+0^2} = \sqrt{10}$

$|\vec{u}+\vec{v}| = |(-2,3,3)| = \sqrt{(-2)^2+3^2+3^2} = \sqrt{22}$.

Note the triangle inequality says $\sqrt{22} \leq \sqrt{14} + \sqrt{10}$ ☐

pf: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be arbitrary

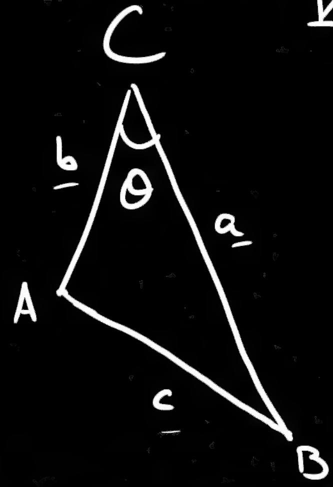
we have

$$\begin{aligned}
|\vec{u}+\vec{v}|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= (\vec{u}+\vec{v})\cdot\vec{u} + (\vec{u}+\vec{v})\cdot\vec{v} \\
&= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \vec{u}\cdot\vec{u} + 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} \\
&= |\vec{u}|^2 + 2(\vec{u}\cdot\vec{v}) + |\vec{v}|^2 \\
&\leq |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2 \\
&= (|\vec{u}| + |\vec{v}|)^2
\end{aligned}$$

Hence $0 \leq |\vec{u}+\vec{v}|$ yields $|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$ as desired. ▨

<u>Recall</u>: Law of Cosines:

Suppose a triangle has



$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

<u>Prop (Angle Formula)</u>: Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$ are at angle θ. Then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$.

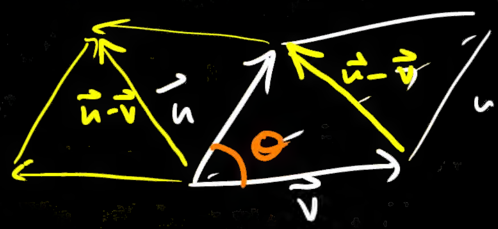
<u>Remark</u>: Typically we use this formula to compute the angle θ; in particular:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \quad \therefore \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$\cos^{-1}$ ← sometimes

WTS: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ Have: Law of Cosines.

pf: Let $\vec{u},\vec{v} \in \mathbb{R}^n$ be arbitrary



$$
\begin{aligned}
|\vec{u}-\vec{v}|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v} \\
&= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= |\vec{u}|^2 - \vec{v}\ \vec{u} - \vec{u}\cdot\vec{v} + |\vec{v}|^2 \\
&= \boxed{|\vec{u}|^2 + |\vec{v}|^2 - 2(\vec{u}\cdot\vec{v})}
\end{aligned}
$$

On the other hand, by the Law of Cosines,

$$|\vec{u}-\vec{v}|^2 = \boxed{|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)}$$

Hence $|\vec{u}|^2 + |\vec{v}|^2 - 2(\vec{u}\cdot\vec{v}) = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$, so we can rearrange this formula to become

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$$ as desired. ☑